A Generalized Jaynes-Cummings Hamiltonian and Supersymmetric Shape-Invariance

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Abstract

A class of shape-invariant bound-state problems which represent two-level systems are introduced. It is shown that the coupled-channel Hamiltonians obtained correspond to the generalization of the Jaynes-Cummings Hamiltonian.

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I. INTRODUCTION

Supersymmetric quantum mechanics [1,2] deals with pairs of Hamiltonians which have the same energy spectra, but different eigenstates. A number of such pairs of Hamiltonians share an integrability condition called shape invariance [3]. Although not all exactly-solvable problems are shape-invariant [4], shape invariance, especially in its algebraic formulation [5–7], is a powerful technique to study exactly-solvable systems.

Supersymmetric quantum mechanics is generally studied in the context of one-dimensional systems. The partner Hamiltonians

$$\hat{H}_1 = \hat{A}^\dagger \hat{A} \tag{1.1a}$$

$$\hat{H}_2 = \hat{A}\hat{A}^\dagger \,, \tag{1.1b}$$

are most readily written in terms of one-dimensional operators

$$\hat{A} \equiv W(x) + \frac{i}{\sqrt{2m}}\hat{p}\,, \tag{1.2a}$$

$$\hat{A}^\dagger \equiv W(x) - \frac{i}{\sqrt{2m}}\hat{p}\,, \tag{1.2b}$$

where $W(x)$ is the superpotential. Attempts were made to generalize supersymmetric quantum mechanics and the concept of shape-invariance beyond one-dimensional and spherically-symmetric three-dimensional problems. These include non-central [8], non-local [9], and periodic [10] potentials; a three-body problem in one-dimension [11] with a three-body force [12]; N-body problem [13]; and coupled-channel problems [14,15]. It is not easy to find exact solutions to these problems. For example, in the coupled-channel case a general shape-invariance is only possible in the limit where the superpotential is separable [15] which corresponds to the well-known sudden approximation in the coupled-channel problem [16]. Our goal in this article is to introduce a class of shape-invariant coupled-channel problems which correspond to the generalization of the Jaynes-Cummings Hamiltonian [17].

II. SHAPE INVARIANCE

The Hamiltonian $\hat{H}_1$ of Eq. (1.1) is called shape-invariant if the condition

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1)\,, \tag{2.1}$$

is satisfied [3]. In this equation a_1 and a_2 represent parameters of the Hamiltonian. The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of the dynamical variables such as position and momentum. As it is written the condition of Eq. (2.1) does not require the Hamiltonian to be one-dimensional, and one does not need to choose the ansatz of Eq. (1.2). In the cases studied so far the parameters a_1 and a_2 are either related by a translation [4,18] or a scaling [19]. Introducing the similarity transformation that replaces a_1 with a_2 in a given operator

$$\hat{T}(a_1)\,\hat{O}(a_1)\,\hat{T}^\dagger(a_1) = \hat{O}(a_2) \tag{2.2}$$

and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \tag{2.3}$$

$$\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1)\,, \tag{2.4}$$

the Hamiltonians of Eq. (1.1) take the forms

$$\hat{H}_1 = \hat{B}_+\hat{B}_-\,. \tag{2.5}$$

and

$$\hat{H}_2 = \hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger\,. \tag{2.6}$$

Using Eq. (2.1) one can also easily prove the commutation relation [5]

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0)\,, \tag{2.7}$$

where we used the identity

$$R(a_n) = \hat{T}(a_1)\, R(a_{n-1})\, \hat{T}^\dagger(a_1)\,, \tag{2.8}$$

valid for any n. The ground state of the Hamiltonian $\hat{H}_1$ satisfies the condition

$$\hat{A} \mid \psi_0\rangle = 0 = \hat{B}_- \mid \psi_0\rangle\,. \tag{2.9}$$

The n-th excited state of $\hat{H}_1$ is given by

$$\mid \psi_n\rangle \sim \left(\hat{B}_+\right)^n \mid \psi_0\rangle \tag{2.10}$$

with the eigenvalue

$$\varepsilon_n = \sum_{k=1}^{n} R(a_k)\,. \tag{2.11}$$

Note that the eigenstate of Eq. (2.10) needs to be suitably normalized. We discuss the normalization of this state in the next section.

III. GENERALIZATION OF THE JAYNES-CUMMINGS HAMILTONIAN

To generalize the Jaynes-Cummings Hamiltonian to general shape-invariant systems we introduce the operator

$$\hat{S} = \sigma_+\hat{A} + \sigma_-\hat{A}^\dagger\,, \tag{3.1}$$

where

$$\sigma_\pm = \frac{1}{2}\left(\sigma_1 \pm i\sigma_2\right)\,, \tag{3.2}$$

with σ_i, with $i = 1$, 2, and 3, being the Pauli matrices and the operators $\hat{A}$ and $\hat{A}^\dagger$ satisfy the shape invariance condition of Eq. (2.1). We search for the eigenstates of $\hat{S}$. It is more convenient to work with the square of this operator, which can be written as

$$\hat{S}^2 = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_-\hat{B}_+ & 0 \\ 0 & \hat{B}_+\hat{B}_- \end{bmatrix} \begin{bmatrix} \hat{T}^\dagger & 0 \\ 0 & \pm 1 \end{bmatrix} . \tag{3.3}$$

Note the freedom of sign choice in this equation, which results in two possible decompositions of $\hat{S}^2$.

We next introduce the states

$$| \, \Psi\rangle_\pm = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} | \, m\rangle \\ | \, n\rangle \end{bmatrix} \tag{3.4}$$

where $| \, m\rangle$ and $| \, n\rangle$ are the abbreviated notation for the states $| \, \psi_n\rangle$ and $| \, \psi_m\rangle$ of Eq. (2.10). Using Eqs. (2.7), (3.3) and (3.4) and the fact that the operator $\hat{T}$ is unitary one gets

$$\begin{aligned} \hat{S}^2 \, | \, \Psi\rangle_\pm &= \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_+\hat{B}_- + R(a_0) & 0 \\ 0 & \hat{B}_+\hat{B}_- \end{bmatrix} \begin{bmatrix} | \, m\rangle \\ | \, n\rangle \end{bmatrix} \\ &= \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \varepsilon_m + R(a_0) & 0 \\ 0 & \varepsilon_n \end{bmatrix} \begin{bmatrix} | \, m\rangle \\ | \, n\rangle \end{bmatrix} . \end{aligned} \tag{3.5}$$

Using Eqs. (2.8) and (2.11) one can write

$$\begin{aligned} \hat{T} \left[\varepsilon_m + R(a_0)\right] \hat{T}^\dagger &= \hat{T} \left[R(a_1) + R(a_2) + \cdots + R(a_m) + R(a_0)\right] \hat{T}^\dagger \\ &= R(a_2) + R(a_3) + \cdots + R(a_{m+1}) + R(a_1) = \varepsilon_{m+1} \, . \end{aligned} \tag{3.6}$$

Hence the states

$$| \, \Psi_m\rangle_\pm = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} | \, m\rangle \\ | \, m+1\rangle \end{bmatrix}, \; m = 0, 1, 2, \cdots \tag{3.7}$$

are the normalized eigenstates of the operator $\hat{S}^2$

$$\hat{S}^2 \, | \, \Psi_m\rangle_\pm = \varepsilon_{m+1} \, | \, \Psi_m\rangle_\pm \, . \tag{3.8}$$

One can also calculate the action of the operator $\hat{S}$ on this state

$$\hat{S} \, | \, \Psi_m\rangle_\pm = \frac{1}{\sqrt{2}} \begin{bmatrix} \pm\hat{T}\hat{B}_- \, | \, m+1\rangle \\ \hat{B}_+ \, | \, m\rangle \end{bmatrix} . \tag{3.9}$$

Introducing the operator [7]

$$\hat{Q}^\dagger = \left(\hat{B}_+\hat{B}_-\right)^{-1/2} \hat{B}_+ \tag{3.10}$$

one can write the normalized eigenstate of $\hat{H}_1$ as

$$| \, m\rangle = \left(\hat{Q}^\dagger\right)^m \, | \, 0\rangle \, . \tag{3.11}$$

Using Eqs. (3.10) and (3.11) one gets

$$\hat{B}_+ \mid m\rangle = \sqrt{\varepsilon_{m+1}} \mid m+1\rangle \,. \tag{3.12}$$

Similarly

$$\begin{aligned} \hat{T}\hat{B}_- \mid m+1\rangle &= \hat{T}\hat{B}_- \frac{1}{\sqrt{\hat{B}_+\hat{B}_-}} \hat{B}_+ \mid m\rangle \\ &= \hat{T}\sqrt{\hat{B}_-\hat{B}_+} \mid m\rangle \\ &= \hat{T}\sqrt{\varepsilon_m + R(a_0)} \mid m\rangle \\ &= \sqrt{\varepsilon_{m+1}}\,\hat{T} \mid m\rangle \,. \end{aligned} \tag{3.13}$$

Using Eqs. (3.12) and (3.13), Eq. (3.9) takes the form

$$\begin{aligned} \hat{S} \mid \Psi_m\rangle_\pm &= \frac{1}{\sqrt{2}}\sqrt{\varepsilon_{m+1}} \begin{bmatrix} \pm\hat{T} \mid m\rangle \\ \mid m+1\rangle \end{bmatrix} \\ &= \pm\sqrt{\varepsilon_{m+1}} \mid \Psi_m\rangle_\pm \,. \end{aligned} \tag{3.14}$$

Eqs. (3.8) and (3.14) indicate that the Hamiltonian

$$\hat{H} = \hat{S}^2 + \sqrt{\hbar\Omega}\,\hat{S} \,, \tag{3.15}$$

where Ω is a constant, has the eigenstates $\mid \Psi_m\rangle_\pm$

$$\hat{H} \mid \Psi_m\rangle_\pm = \left(\varepsilon_{m+1} \pm \sqrt{\hbar\Omega}\sqrt{\varepsilon_{m+1}}\right) \mid \Psi_m\rangle_\pm \tag{3.16}$$

with the exception of the ground state. It is easy to show that the ground state is

$$\mid \Psi_0\rangle = \begin{bmatrix} 0 \\ \mid 0\rangle \end{bmatrix}, \tag{3.17}$$

with eigenvalue 0. To emphasize the structure of Eq. (3.16) as the generalized Jaynes-Cummings Hamiltonian we rewrite it as

$$\hat{H} = \hat{A}^\dagger\hat{A} + \frac{1}{2}\left[\hat{A}, \hat{A}^\dagger\right](\sigma_3 + 1) + \sqrt{\hbar\Omega}\left(\sigma_+\hat{A} + \sigma_-\hat{A}^\dagger\right) \,. \tag{3.18}$$

When $\hat{A}$ describes the annihilation operator for the harmonic oscillator, $\left[\hat{A}, \hat{A}^\dagger\right] = \hbar\omega$, where ω is the oscillator frequency. In this case Eq. (3.18) reduces to the standard Jaynes-Cummings Hamiltonian.

When $\hat{A}^\dagger\hat{A}$ describes the Morse Hamiltonian, Eq. (3.18) takes the form

$$\begin{aligned} \hat{H} = {} & \frac{\hat{p}^2}{2M} + V_0\left(e^{-2\lambda x} - 2e^{-\lambda x}\right) + \sqrt{V_0}\frac{\hbar\lambda}{\sqrt{2M}}(\sigma_3 + 1)\,e^{-\lambda x} \\ & + \sqrt{\hbar\Omega V_0}\left[\sigma_1\left(1 - \frac{\hbar\lambda}{2\sqrt{2MV_0}} - e^{-\lambda x}\right) - \sigma_2\frac{\hat{p}}{\sqrt{2MV_0}}\right] \end{aligned} \tag{3.19}$$

with the energy eigenvalues

$$E_m = \sqrt{V_0}\frac{\hbar\lambda}{\sqrt{2M}}(m+1)\left[2-\frac{\hbar\lambda}{\sqrt{2MV_0}}(m+2)\right]$$
$$\pm\left\{\hbar\Omega\sqrt{V_0}\frac{\hbar\lambda}{\sqrt{2M}}(m+1)\left[2-\frac{\hbar\lambda}{\sqrt{2MV_0}}(m+2)\right]\right\}^{\frac{1}{2}}. \qquad (3.20)$$

Both harmonic oscillator and Morse potential are shape-invariant potentials where parameters are related by a translation. It is also straightforward to use those shape-invariant potentials where the parameters are related by a scaling [19] in writing down Eq. (3.18).

IV. CONCLUSIONS

In this article we introduced a class of shape-invariant bound-state problems which represent two-level systems. The corresponding coupled-channel Hamiltonians generalize the Jaynes-Cummings Hamiltonian. If we take $\hat{H}_1$ to be the simplest shape-invariant system, namely the harmonic oscillator, our Hamiltonian, Eq. (3.18), reduces to the standard Jaynes-Cummings Hamiltonian, which has been extensively used to model a single field mode on resonance with atomic transitions.

In this article we only addressed generalization of the Jaynes-Cummings model to other shape-invariant bound state systems. Supersymmetric quantum mechanics has been applied to alpha particle [20] and Coulomb [21] scattering problems. More recently shape-invariance was utilized to calculate quantum tunneling probabilities [22]. It may be possible to generalize our results to such continuum problems. Such an investigation will be deferred to a later publication.

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REFERENCES

[1] E. Witten, Nucl. Phys. B **185**, 513 (1981).

[2] For a recent review see F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251**, 267 (1995).

[3] L. Gendenshtein, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 299 (1983) [JETP Lett. **38**, 356 (1983)].

[4] F. Cooper, J. N. Ginocchio and A. Khare, Phys. Rev. D **36**, 2458 (1987).

[5] A. B. Balantekin, Phys. Rev. A **57**, 4188 (1998).

[6] S. Chaturvedi, R. Dutt, A. Gangopadhyay, P. Panigrahi, C. Rasinariu, and U. Sukhatme, Phys. Lett. A **248**, 109 (1998).

[7] A. B. Balantekin, M. A. Cândido Ribeiro, and A. N. F. Aleixo, J. Phys. A: Math. Gen. **32**, 2785 (1999).

[8] R. Dutt, A. Gangopadhyay, and U. Sukhatme, Am. J. Phys. **65**, 400 (1997).

[9] J.-Y. Choi and S.-I. Hong, Phys. Rev. A **60**, 796 (1999).

[10] G. Dunne and J. Feinberg, Phys. Rev D **57**, 1271 (1998).

[11] D. Z. Freedman and P. F. Mende, Nucl. Phys. B **344**, 317 (1990).

[12] A. Khare and R. K. Bhaduri, J. Phys. A: Math. Gen. **27**, 2213 (1994).

[13] P. K. Ghosh, A. Khare, and M. Sivakumar, Phys. Rev. A **58**, 821 (1998).

[14] R. D. Amado, F. Cannata, and J.-P. Dedonder, Phys. Rev. A **38**, 3797 (1988); Int. J. Mod. Phys. A **5**, 3401 (1990).

[15] T. K. Das and B. Chakrabarti, J. Phys. A: Math. Gen. **32**, 2387 (1999).

[16] A. B. Balantekin and N. Takigawa, Rev. Mod. Phys. **70**, 77 (1998).

[17] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

[18] C. Chuan, J. Phys. A: Math. Gen. **24**, L1165 (1991).

[19] A. Khare and U. Sukhatme, J. Phys. A: Math Gen. **26**, L901 (1993); D. Barclay *et al.*, Phys. Rev. A **48**, 2786 (1993).

[20] D. Baye, Phys. Rev. Lett. **58**, 2738 (1987).

[21] R. Amado, Phys. Rev. A **37**, 2277 (1988).

[22] A. N. F. Aleixo, A. B. Balantekin, and M. A. Cândido Ribeiro, J. Phys. A: Math. Gen., in press (quant-ph/9910051).